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ON A PHYSICAL GEOMETRY AND NEW STATISTICAL CONNECTIONS

BY

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Abstract. An action generated by a 2×2 real matrix is described. It is a simply transitive action, and characterizes a gauge proper, connected to the matrices representing stresses in a continuum. More than this, such a gauge is also related to a possible superstatistic of the Cauchy type. The geometry related to this gauge is a three-dimensional generalization of the plane hyperbolic geometry, from which it can be actually obtained by a Bäcklund transformation involving the gauge angle. In real terms this geometry is a Lorentz three-dimensional geometry. An interesting physical interpretation results for the gauge angle. Moreover, strong connections with the Kuznetsov model for tumor growth can be established.

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1. The Binary Cubic: Meaning and Notations

Cubic equation is the basis for constitutive laws as we accept them today, inasmuch as it allows us to algebraically characterize a 3×3 matrix, regardless if it is a tensor or not. The following treatment refers to the most general form of cubic equation having *real coefficients*. These coefficients are orthogonal invariants in the case of a 3×3 matrix which is tensor with respect to orthogonal group of space, and therefore they bear physical meanings. We accept the extension of these meanings in general, *i.e.* regardless if the matrix is tensor or not. The cubic equation will be written, for convenience, in the so-called *binomial form* as

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \tag{1}$$

Here we assume that the coefficients a_k are real, displaying by a_0 the possibility of adjusting them by an arbitrary factor on account of the known arbitrariness allowed by the relations between the roots and the coefficients of an algebraic equation.

2. The Algebra Related to Cubic Equation

The central problem related to Eq. (1) is, of course, that of finding its roots. These are the eigenvalues of the corresponding matrix, and are usually supposed to be accessible to measurement. There are many methods for the general solution of this problem, conveniently coping with the purpose which that solution is serving (Cocolicchio and Viggiano, 2000; Nickalls, 1993). All of the methods of solution are however centered on reducing it to that of a quadratic equation and, for physics purposes, we think it is worthwhile revealing what this actually entails. The most general theory behind the procedure has been established by Sylvester (Burnside and Panton, 1960) and amounts to putting the Eq. (1) in the form of a sum of two perfect cubes:

$$\beta_1(x-\alpha_1)^3 + \beta_2(x-\alpha_2)^3 = 0 \tag{2}$$

In this case the equation can be easily solved to give

$$x_j = \frac{\alpha_1 + \alpha_2\varepsilon_j k}{1+\varepsilon_j k}; \quad j = 1,2,3 \tag{3}$$

where we denoted

$$k^3 = \frac{\beta_2}{\beta_1}, \quad \varepsilon^3 = 1 \tag{4}$$

i.e. ε_j are the cubic roots of the unity:

$$\varepsilon_1 = 1; \quad \varepsilon_2 = \frac{-1 + i\sqrt{3}}{2}; \quad \varepsilon_3 = \frac{-1 - i\sqrt{3}}{2}$$

The problem of solving the cubic equation is thus translated into that of finding the quantities α_1, α_2, β_1, β_2 from Eq. (2) as functions of the coefficients a_0, a_1, a_2, a_3, which are usually related to physical situations. This can be done as follows: identifying the Eqs. (1) and (2), gives the following system of equations

$$\begin{aligned} \beta_1 + \beta_2 = a_0, \quad \beta_1\alpha_1 + \beta_2\alpha_2 = -a_1 \\ \beta_1\alpha_1^2 + \beta_2\alpha_2^2 = a_2, \quad \beta_1\alpha_1^3 + \beta_2\alpha_2^3 = -a_3 \end{aligned} \tag{5}$$

Now, we always may assume that α_1 and α_2 are the roots of a quadratic equation, which we write in the form:

$$b_0\alpha^2 + b_1\alpha + b_2 = 0$$

This equation entails the natural identities

$$\begin{aligned} b_0\alpha_1^2 + b_1\alpha_1 + b_2 = 0 \\ b_0\alpha_2^2 + b_1\alpha_2 + b_2 = 0 \end{aligned} \tag{6}$$

which we try to put in relation with the system of Eqs. (5). For instance, we can obtain one equation by multiplying the first of the Eqs. (6) by β_1 the second one by β_2 and then adding the results. The coefficients of b_0, b_1, b_2 in the new equation are given by a_0, a_1, a_2 from (5). Likewise, another equation may be obtained when multiplying the first of the Eqs. (6) by $\beta_1\alpha_1$, the second one by $\beta_2\alpha_2$ and then add the resulting equations. The end result for this procedure is the following system of equations for b_0, b_1, b_2:

$$\begin{aligned} a_2b_0 - a_1b_1 + a_0b_2 = 0 \\ a_3b_0 - a_2b_1 + a_1b_2 = 0 \end{aligned}$$

This system has the solution defined, up to an arbitrary factor, by the equations

$$\frac{b_0}{a_0a_2 - a_1^2} = \frac{b_1}{a_0a_3 - a_1a_2} = \frac{b_2}{a_1a_3 - a_2^2}$$

showing that α_1 and α_2 from (2) are the roots of a quadratic equation:

$$(a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)x + (a_1a_3 - a_2^2) = 0 \tag{7}$$

called *Hessian* associated to the cubic Eq. (1). Now we must find β_1 and β_2 from (2). For this we can use any pair from the four Eqs. (5). The result is the same up to a factor. Because we actually need only the ratio β_2/β_1, we can however use another, more direct method (Vodă, 1987), having the advantage to exhibit straightforwardly the algebraic nature of β_1 and β_2. Namely, denoting the cubic from Eq. (1) by f(x), and using the Eq. (2), we find

$$f(\alpha_1) = \beta_2(\alpha_1 - \alpha_2)^3, \quad f(\alpha_2) = -\beta_1(\alpha_1 - \alpha_2)^3$$

whence the ratio between β_1 and β_2 is given by equation

$$\frac{\beta_2}{\beta_1} = -\frac{f(\alpha_1)}{f(\alpha_2)} \tag{8}$$

Therefore β_1 and β_2 are of the same algebraic nature as α_1 and α_2, because the coefficients of the cubic are real. Thus, the solution of a cubic equation can, indeed, be reduced to that of a quadratic equation. Here the quadratic in question is the Hessian of the cubic. In some other methods of solution we might have some other quadratic, but it will still be in close relationship with the Hessian.

The relation between cubic and its Hessian can be summarized by the following three general theorems:

Theorem 1. *If a cubic has the Hessian a perfect square, then such a cubic contains the Hessian as a factor.*

Theorem 2. *If the Hessian of a cubic equation has distinct roots then the cubic itself has distinct roots. There are thus two cases:*

a) if the Hessian has real roots, then the cubic itself has one real and two complex roots.

b) if the Hessian has complex roots, then the cubic itself has real roots

Theorem 3. *If a cubic is a perfect cube, then it has a null Hessian. Reciprocally, if a cubic has a null Hessian then it is a perfect cube.*

It is therefore important to introduce the distinguished quantity, playing an essential role in the theory of cubic equations. This quantity is the *discriminant* of the Hessian of a cubic, also called the *discriminant of the cubic* itself. In view of Eq. (7) it is, obviously, given by

$$\Delta = (a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) \tag{9}$$

and deserves this name because, like in the case of the quadratic equation, it decides the algebraic nature of the roots of cubic equation. For instance, the second of the theorems above can be directly proved just considering the form (3) of the roots of the cubic. The first theorem, and the third, can be proved using Eq. (9) in the special cases to which they are referring.

3. An External Gauge Factor

As one can see from the above presentation, the Hessian of a cubic is a key tool in constructing the roots of that cubic. Sometimes, in practical problems, the physical principles even allow us to know the Hessian before we know the cubic itself, and in such cases we need to figure out the corresponding cubic, more specifically to construct the roots of this cubic. One can guess that such situations currently occur in the case of nuclear matter, and this should be indeed the case if judged from a group-theoretical point of view.

The previous development of the theory of cubic equations shows that, given the roots of the Hessian only, one cannot know the corresponding cubic equation without ambiguity. In fact the Eqs. (2) and (3) show that *to a given Hessian there corresponds a one-parameter family of triplets of numbers*, each one of these triplets representing a given cubic equation. This indetermination is independent of the known property of indetermination allowed by the relations between roots and coefficients. As a matter of fact, it is even deeper than the Eq. (2) shows it, in the sense that the ratio k, which by Eqs. (4) and (8) depends only on the quantities related to the cubic equation, may hide in itself an external phase completely independent of the cubic equation – a gauge phase.

This observation and the algebraic proof that follows are due to Dan Barbilian (Barbilian, 1971). In order to better grasp the nature of this problem, we use the following identity between cubic itself (f), its Hessian (H) and its Jacobian (T):

$$4H^3 = \Delta \cdot f^2 - T^2 \tag{10}$$

The expression in right hand side of this equation can be decomposed into two factors each of the third degree, because the cubic and its Jacobian are prime with respect to each other. On the other hand, the left hand side is a product of two perfect cubes, because the Hessian is a quadratic polynomial. The identity (10) then shows that each factor of the right hand side is proportional to a factor of the expression from the left hand side, and this proportionality can be taken in two ways at will. However, for a fixed choice between those two ways, the proportionality factors should be reciprocal to one another. Indeed, the Hessian can be factorized in infinite many ways as

$$H = mU \cdot \frac{1}{m} V$$

where U and V are first degree binomials and ‘m’ is any nonzero number. Thus the identity (10) can be written as the system

$$\sqrt{\Delta} f + T = 2m^3 U^3, \quad \sqrt{\Delta} f - T = 2m^{-3} V^3 \tag{11}$$

Adding these expressions, gives the result (2) only in a slightly different form, showing clearly where the external arbitrariness comes into play:

$$\sqrt{\Delta}f = m^3U^3 + m^{-3}V^3 \tag{12}$$

One can further decompose the right hand side here into linear factors, to the effect that (12) becomes

$$\sqrt{\Delta}f = (mU + m^{-1}V)\cdot(\theta mU + \theta^2 m^{-1}V)\cdot(\theta^2 mU + \theta m^{-1}V)$$

This form allows us to find the roots of the cubic equation f = 0 in the form given by Eq. (2) with $k \equiv m^{-2}$. In case the roots are all real, k must be complex unimodular as before. For the sake of completeness, we mention that the Jacobian of a cubic can be also obtained from (11) as a difference of cubes:

$$\begin{aligned} T &= m^3U^3 - m^{-3}V^3 \\ &= (mU - m^{-1}V)\cdot(\theta mU - \theta^2 m^{-1}V)\cdot(\theta^2 mU - \theta m^{-1}V) \end{aligned}$$

This shows that the roots of Jacobian are of the same algebraical nature as the roots of the cubic itself. In formula (3) we do not have to change but the sign in both the denominator and numerator in order to get the roots of the corresponding Jacobian. This discussion also shows that the form (3) of the roots of a cubic equation is valid independently of the nature of the roots.

4. A Physical Interpretation

It is now important to give a *physical interpretation* for the external factor k occurring when one wants to construct the cubic given its Hessian. For this we will consider the case where the cubic has *real* roots, i.e. k is complex of unit modulus. The Eq. (3) for the roots can be written as (Barbilian, 1938).

$$x_1 = \frac{h + h^*\cdot k}{1+k}, \quad x_2 = \frac{h + \varepsilon\cdot h^*\cdot k}{1+\varepsilon\cdot k}, \quad x_3 = \frac{h + \varepsilon^2\cdot h^*\cdot k}{1+\varepsilon^2\cdot k} \tag{13}$$

with h, h^* – the roots of Hessian and $\varepsilon \equiv (-1 + i\sqrt{3})/2$ the cubic root of unity ($i \equiv \sqrt{(-1)}$). Now consider the *vector* of components x_1, x_2, x_3. This is a ‘vector’ indeed, but with respect to a special group to be mentioned later. For now, it just happens to represent a real space situation when the three roots are the principal values of a *symmetric matrix*. We are certainly correct in using this image, at least in a limited way, for there is a space reference frame we can construct in every point of space where the symmetric matrix is defined. This is given by three special orthogonal vectors – the principal directions of the symmetric matrix in question. Thus the principal values of such a matrix can be arranged in the column matrix

$$\left|x\right\rangle \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{14}$$

which is plainly a vector in matrix representation. Indeed each principal value of the matrix can be interpreted as the component of the vector along the corresponding principal direction.

We can decompose the vector from (14) with respect to a plane cutting the axes of reference frame in the points situated at unit distance from origin. In engineering applications such a plane is called *octahedral plane*, for it represents one of the faces of an octahedron in space. Assuming therefore the situation in the first octant of our reference frame, the *normal component* of the vector (14) on this plane is, with an obvious notation for transposed vectors, given by

$$\left|x_n\right\rangle \equiv \left|n\right\rangle\left\langle n\right|x\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}(1,1,1) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{x_1 + x_2 + x_3}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The in-plane (*tangential*) component of (14) is then given by

$$\left|x_t\right\rangle \equiv \left|x\right\rangle - \left|x_n\right\rangle = \frac{1}{3}\begin{pmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_3 - x_1 \\ 2x_3 - x_1 - x_2 \end{pmatrix}$$

It is this last vector, usually called *octahedral shear vector* in engineering applications – its components are given by the eigenvalues of the so-called deviator of the original matrix – which allows us to interpret the complex number k externally introduced. Namely, the Sylvester form (2) of our cubic allows us to identify its binomial coefficients in terms of the quantities h, h^* and k, up to an arbitrary factor, as

$$\begin{aligned} &a_0 = 1 + k^3, \quad a_1 = -(h + h^* \cdot k^3), \\ &a_2 = h^2 + h^{*2} \cdot k^3, \quad a_3 = -(h^3 + h^{*3} \cdot k^3) \end{aligned} \tag{15}$$

From this we have right away

$$\frac{1}{3}\sum x_1 = \frac{h + h^* \cdot k^3}{1+k^3} \quad \therefore \quad |x_t\rangle = \frac{(h-h^*)k}{1+k^3}\begin{pmatrix} k-1 \\ \omega(\omega k-1) \\ \omega^2(\omega^2 k-1)\end{pmatrix} \tag{16}$$

Now take as reference in the octahedral plane the vector corresponding to k = 1, when the roots of the cubic are exclusively determined by the roots of its Hessian, therefore with no arbitrariness whatsoever:

$$|x_t^0\rangle = \frac{(h-h^*)}{2}\begin{pmatrix} 0 \\ \omega(\omega-1) \\ \omega^2(\omega^2-1)\end{pmatrix} \tag{17}$$

Then calculate the angle of the generic tangential vector with respect to this one by the well-known formula:

$$\cos\phi \equiv \frac{\langle x_t^0 | x_t\rangle}{\sqrt{\langle x_t^0 | x_t^0\rangle\langle x_t | x_t\rangle}} \tag{18}$$

Using Eqs. (16) and (17) we get

$$\langle x_t^0 | x_t\rangle = -\frac{3(h-h^*)^2 k}{2(1+k^3)}(k+1),$$

$$\langle x_t | x_t\rangle = -6k\frac{(h-h^*)^2 k^2}{(1+k^3)^2}, \quad \langle x_t^0 | x_t^0\rangle = -6\frac{(h-h^*)^2}{2^2}$$

so that Eq. (18) becomes

$$\cos\phi \equiv \frac{1}{2}(\sqrt{k} + \frac{1}{\sqrt{k}}) \tag{19}$$

We conclude that, indeed, knowing the Hessian does not determine the cubic uniquely. In the case of a cubic with real roots the Hessian actually determines a family of cubics whose roots are defined as a one parameter family. *The parameter of this family is given by the angle of orientation of the corresponding octahedral shear vector in the octahedral plane*.

5. Novozhilov's Statistics and the Specification of Hessian

The advantage of this last approach to characterizing the cubic equation is mostly *physical*: as mentioned before, more often than not in physics and engineering problems we have to do with quantities qualifying as coefficients of the Hessian of a cubic function. In this case one can rightfully say that the angle ϕ represents, properly speaking, *a gauge freedom*. Until a proper geometrization of this statement, let us see however how the Hessian can be built, by using again an example from the deformation theory (Novozhilov, 1952). As the Manton's geometrization of Skyrme theory allows us to infer, this construction should be proper in the realm of nuclear matter. On the other hand, it is perhaps worth mentioning that the deformation of matter, in general, can be properly represented, from a physical point of view, as a gauge process.

The statement that we can measure a certain multidimensional physical magnitude in a certain point of space envisions a highly idealized situation. First of all we are not always able to simultaneously measure multiple physical quantities, in view of the fact that these may interact in such a way that their measurements are mutually exclusive. A well known example is the one of the conjugated variables in quantum mechanics. Nevertheless, we shouldn't go that far with the imagination, for the most obvious example is in the very deformation of a continuum. Indeed, while from experimental point of view, we can afford adequate pieces of matter to represent ideal states of deformation as close as possible to the standards we desire, inside a continuum the situation changes drastically. One cannot state that in a certain point of that continuum there is a precise state of deformation of a kind or another. The most we can think of is a mixture of such states, and even that is a highly idealized situation, for we don't know how the states of deformation coexist with each other. But, in the cases where the deformation is thought in terms of 3×3 matrices, the reason can always be conducted along the lines that follow, indicated by Novozhilov.

A matrix quantity defined in a point in space cannot be measured but by its intensities along directions and in planes through that point. The values of these intensities obviously vary with the direction and plan of measurement. However, in a continuum, one can assume that, at least in certain conditions of isotropy, the local manifestation of a matrix quantity is a certain average over all of the possible directions and planes through a point. When the matrix is a symmetric tensor, as one currently assumes in the theory of deformations, and furthermore, when one admits a uniform distribution of all directions and planes in space, the averages over directions and planes can be given quite easily.

If **x** is our matrix, having the eigenvalues $x_{1,2,3}$, then the intensity along a certain direction given by the unit vector $\hat{n}$, can be calculated with the formula

$$x_n \equiv x_{ij} n^i n^j = x_1 (n^1)^2 + x_2 (n^2)^2 + x_3 (n^3)^2 \tag{20}$$

where $n^{1,2,3}$ are the components of the unit vector of direction in the proper system of eigendirections of **x**. Then we can figure out that, in each one of the space points, a continuum can be characterized by an average of this quantity over the unit sphere. Representing the components of $\hat{n}$ in terms of spherical angles as usual: $\sin\theta\cos\varphi$, $\sin\theta\sin\varphi$, $\cos\theta$, one can assume therefore that the continuum exhibits in any point the mean

$$\bar{x}_n = \frac{1}{4\pi}\oiint_{\text{Unit Sphere}} x_n \sin\theta d\theta d\varphi$$

Performing this operation in (20), gives the well known value

$$\bar{x}_n = \frac{x_1 + x_2 + x_3}{3} \tag{21}$$

On the other hand, if $\hat{n}$ is the normal to a plane through a point inside a continuum, we can calculate the intensity of **x** on this plane, according to the formula representing Pythagoras' theorem

$$x_t^2 \equiv (\mathbf{x}^2)_{ij} n^i n^j - (x_{ij} n^i n^j)^2 \tag{22}$$

Using the same procedure of averaging, we can find the point average of this quantity in a point of the continuum:

$$\overline{x_t^2} = \frac{1}{15}\left\{(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2\right\} \tag{23}$$

It is therefore to be expected that, in a continuum without inhomogeneities, when it comes to the measurement of a tensor, we only have at our disposal the quantities (21) and (23), in any of its points. And from these two quantities we ought to construct the eigenvalues of the tensor. Obviously then, the tensor is not uniquely defined. Even if the eigenvalues would be at our disposal, we still would have at least the arbitrariness of space rotations in the definition of a tensor. However they are not at our disposal, and we ought to construct them first, using just the quantities (21) and (23).

In order to do this, we use the previous phase freedom, whereby the case of *null phase* is well determined by these two quantities. Specifically, in that case, which we take as a reference case, we have for the roots of Hessian

$$u = \bar{x}_n;\quad v = \sqrt{\frac{5}{6}\overline{x_t^2}};\quad h \equiv u + iv \tag{24}$$

One can see that the imaginary part of h is simply proportional with the magnitude of the shear vector in the octahedral plane. Therefore the orientation of this shear vector in octahedral plane is arbitrary, and *this is our gauge* *freedom*. With these results, the formulas from Eq. (13) define a one parameter family of cubics, corresponding to the rotations of the shear vector in its octahedral plane.

6. Barbilian's Differential Geometry

However, having now concentrated on the pure mathematical side of the problem, we ought to consider one further algebraic advantage: the values of variables h, h^* and k can be 'scanned' by a simply transitive continuous group with real parameters. Therefore the gauge freedom is way richer than the arbitrary phase lets to be seen. This group has been exhibited for the first time by Dan Barbilian (Barbilian, 1938) with the occasion of a study of the Riemannian space associated with the previous family of cubics. We will briefly review Barbilian's theory insisting on some particular technical points necessary for our reference. The basis of approach is the fact that the simply transitive group with real parameters (Baker, 1901).

$$x_k \quad \leftrightarrow \quad \frac{ax_k + b}{cx_k + d}, \quad a, b, c, d \in R$$

where x_k are the cubic roots previously discussed, induces a simply transitive group for the quantities h, h^* and k, whose action is:

$$h \quad \leftrightarrow \quad \frac{ah + b}{ch + d}, \quad h^* \quad \leftrightarrow \quad \frac{ah^* + b}{ch^* + d}, \quad k \quad \leftrightarrow \quad \frac{ch^* + d}{ch + d} \cdot k$$

which will be called *Barbilian group*. The structure of this group is typical of a SL(2, R) one, which we take in the standard form

$$[B_1, B_2] = B_1, \quad [B_2, B_3] = B_3, \quad [B_3, B_1] = -2B_2 \tag{25}$$

where B_k are the infinitesimal generators of the group. Because the group is simply transitive these generators can be easily found as the components of the *Cartan frame* (Fels and Olver, 1998; Fels and Olver, 1999) from the formula

$$\begin{aligned} d(f) &\equiv \sum \frac{\partial f}{\partial x^k} dx^k = \\ &= \left[\omega^1 \left(h^2 \frac{\partial}{\partial h} + h^{*2} \frac{\partial}{\partial h^*} + \left(h - h^* \right) k \frac{\partial}{\partial k} \right) + 2\omega^2 \left(h \frac{\partial}{\partial h} + h^* \frac{\partial}{\partial \bar{\bar{h}}} \right) + \omega^3 \left(\frac{\partial}{\partial h} + \frac{\partial}{\partial h^*} \right) \right] (f) \end{aligned} \tag{26}$$

where ω^k are the components of the *Cartan coframe* to be found from the system

$$dh = \omega^1 h^2 + 2\omega^2 h + \omega^3$$
$$dh^* = \omega^1 h^{*2} + 2\omega^2 h^* + \omega^3$$
$$dk = \omega^1 k(h - h^*)$$

Thus we have immediately both the infinitesimal generators and the coframe by identifying the right hand side of Eq. (26) with the standard dot product of SL(2, R) algebra:

$$\omega^1 B_3 + \omega^3 B_1 - 2\omega^2 B_2$$

so that

$$B_1 = \frac{\partial}{\partial h} + \frac{\partial}{\partial h^*}, \qquad B_2 = h\frac{\partial}{\partial h} + h^*\frac{\partial}{\partial h^*}$$
$$B_3 = h^2\frac{\partial}{\partial h} + h^{*2}\frac{\partial}{\partial h^*} + (h - h^*)k\frac{\partial}{\partial k} \tag{27}$$

and

$$\omega^1 = \frac{dk}{(h - h^*)k}, \qquad 2\omega^2 = \frac{dh - dh^*}{h - h^*} - \frac{h + h^*}{h - h^*}\frac{dk}{k}$$
$$\omega^3 = \frac{hdh^* - h^*dh}{h - h^*} + \frac{hh^*}{h - h^*}\frac{dk}{k}$$

In real terms: $h = u + iv$, $k = e^{i\phi}$, these last equations can be written as

$$B_1 = \frac{\partial}{\partial u}, \qquad B_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \qquad B_3 = (u^2 - v^2)\frac{\partial}{\partial u} + 2uv\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial \phi}$$
$$\omega^1 = \frac{d\phi}{2v} \qquad \omega^2 = \frac{dv}{v} - \frac{u}{v}d\phi \qquad \omega^3 = \frac{u^2 + v^2}{2v}d\phi + \frac{vdu - udv}{v} \tag{28}$$

Mention should be made that, in his original paper, Barbilian does not work with the above differential forms but with the *absolute invariant differentials*

$$\omega^1 = \frac{dh}{(h - h^*)k}, \qquad \omega^2 = -i\left(\frac{dk}{k} - \frac{dh + dh^*}{h - h^*}\right), \qquad \omega^3 = -\frac{kdh^*}{h - h^*} \tag{29}$$

or, in real terms, exhibiting a three-dimensional Lorentz structure of this space

$$\Omega^1 \equiv \omega^2 = d\phi + \frac{du}{v},$$
$$\Omega^2 = \cos\phi\frac{du}{v} + \sin\phi\frac{dv}{v}, \qquad \Omega^3 = -\sin\phi\frac{du}{v} + \cos\phi\frac{dv}{v} \tag{30}$$

The advantage of this representation is that it makes obvious the connection with the Poincaré representation of the Lobachevsky plane. Indeed, the metric here is

$$-(\Omega^1)^2+(\Omega^2)^2+(\Omega^3)^2=-(d\phi+\frac{du}{v})^2+\frac{(du)^2+(dv)^2}{v^2} \tag{31}$$

This metric reduces to that of Poincaré in case where $\Omega^1 = 0$ which, as Barbilian noticed, defines the variable ϕ as the 'angle of parallelism' of the hyperbolic plane (the connection). In fact, recalling that in modern terms (du/v) represents the connection form of the hyperbolic plane (Flanders, 1989), the Eqs. (30) then represent a general Bäcklund transformation in that plane (Sasaki, 1979; Rogers and Schief, 2002).

7. The Apolar Transport of Cubics

In view of the importance that we revealed for the geometry of Lobachevsky in the classical Kepler problem, it becomes also important to know the meaning of the condition $\Omega^1 = 0$ for a family of cubic equations. It turns out that it expresses the so-called *apolar transport* of cubics (Barbilian, 1938), whereby a certain cubic evolves in such a way that its roots remain always in a harmonic progression. This transport is defined by the condition that any root of the 'transported' cubic is in a harmonic relation with any root of the 'original' cubic, with respect to the other two remaining roots of the original cubic:

$$\frac{y_1-x_j}{y_1-x_k}:\frac{x_i-x_j}{x_i-x_k}=-1; \quad i\neq j\neq k\neq i \tag{32}$$

in all positive permutations of the indices i, j, k and for every l. Therefore each new root (y_l) and each of the corresponding old ones (x_i), are in harmonic range with respect to the other two old roots (x_j, x_k). Then it can be proved that the conditions from equation (32) boil down to the vanishing of the bilinear invariant of the two cubics, analogous to the bilinear invariant of the quadratics:

$$a_3b_0-3a_2b_1+3a_1b_2-a_0b_3 \tag{33}$$

Here a_m denote the coefficients of the original cubic, while b_m denote the coefficients of the transported cubic. Obviously, this invariant is analogous to the one from the case of two quadratics, whose vanishing expresses the fact that their roots are in harmonic sequence. The geometry related to this invariant is century old (see (Burnside and Panton, 1960)) and Dan Barbilian seemed particularly fond of it (Barbilian, 1935), for he elaborated for a long while on its different aspects, especially related to the geometry of the triangle. As the triangle comes nowadays in relation with the construction of skyrmions from instantons (Atiyah and Manton, 1989; Manton, 1989), from a point of view closely related to its geometry, it is therefore worth considering this connection, which turns out to be strictly related to the physics of continua.

Now, if the two cubics are infinitesimally close, then the condition of their transport by involution reduces to

$$a_3 da_0 - 3a_2 da_1 + 3a_1 da_2 - a_0 da_3 = 0 \tag{34}$$

Using here the Eqs. (15) above for the coefficients, the condition of apolar transport of the cubics amounts to

$$-(h-h^*)^3 \cdot k^3 \left(\frac{dk}{k} - \frac{dh + dh^*}{h - h^*} \right) = 0$$

As the cubics are asumed to have distinct roots, this condition is satisfied if, and only if, the differential form Ω^1 is null. Therefore the parallel transport of the hyperbolic plane actually represents the apolar transport of the cubics.

This way, the vectors represented by the real eigenvalues of a certain matrix, have a sure physical interpretation in the framework of the classical theory of Kepler motion. Assume, for the sake of exemplification a hydrogen atom: it is described by a single Kepler motion. We have seen that the nuclear matter in such representation can be characterized by a complex number depending on the eccentricity and the orientation of the orbit. This complex number can be assumed to represent a particular state of stress inside nucleus, given by Eqs. (24) above. The eigenvalues of stress are then given by Barbilian formulas using the gauge freedom. However, insofar as they are supposed to be measured values themselves, they reveal an outstanding meaning of the root of Hessian: it is the (complex) parameter of a Cauchy distribution.

8. Peter McCullagh's Observation on Cauchy Statistics

Peter McCullagh has noticed a curious property of the one-dimensional Cauchy distribution, which is related to the benefit of a complex parameterization of this distribution (McCullagh, 1996). The parameters of a statistical distribution are usually taken as real, but McCullagh shows a clear advantage of representing them in a complex form, at least when it comes to Cauchy distribution. He starts with the fact that this distribution for a single variate X can be written in the form

$$f_X(x \mid \theta) = \frac{|\theta_2|}{\pi |x - \theta|^2}; \quad \theta \equiv \theta_1 + i\theta_2 \tag{35}$$

where θ is the 'complex parameter' of the distribution. The real part of this parameter gives the location of data, while the imaginary part roughly characterizes the spread of the distribution. One knows that this class of distributions is closed with respect to the homographic transformation of the variable: any linear fractional transform of X has also a Cauchy distribution. But the complex representation of the parameter brings to light one of the most important consequences of this theorem: if X belongs to the Cauchy class with the complex parameter θ, *i.e.* symbolically $X \approx C(\theta)$, then we have

$$\frac{aX+b}{cX+d} \approx C\left(\frac{a\theta+b}{c\theta+d}\right) \tag{36}$$

This property allows us to give *efficient estimators* for the complex parameter θ, based on the principle of maximum likelihood.

As a rule, the likelihood function used in estimations is simply the product of the values of the probability density for the different measured values of X. In taking the maximum likelihood with respect to parameters, it would be therefore appropriate to work with the logarithm of the likelihood, and this is what practically happens. For instance if one measures two values of X having the probability density (35), say x_1 and x_2, the likelihood function constructed based on this information is simply:

$$L(\theta \mid x_1, x_2) = \frac{\theta_2^2}{\pi^2 |x_1-\theta|^2 |x_2-\theta|^2} \tag{37}$$

The likelihood is maximum with respect to θ when the derivatives of this function with respect to θ_1 and θ_2 are null. In terms of the log-likelihood, which is a lot easier to handle, we then have:

$$\frac{\partial}{\partial\theta_1}\ln L(\theta \mid x_1, x_2) = \frac{\partial}{\partial\theta_2}\ln L(\theta \mid x_1, x_2) = 0 \tag{38}$$

In view of the fact that

$$\ln L(\theta \mid x_1, x_2) = -2\ln\pi + 2\ln|\theta_2| - \sum_i \ln(x_i-\theta) - \sum_i \ln(x_i-\theta^*) \tag{39}$$

where the sumation extends over the two measured values and a star denotes complex conjugation, the two Eq. (38) become:

$$\sum_i \frac{1}{x_i-\theta} + \sum_i \frac{1}{x_i-\theta^*} = 0; \quad \frac{2}{i\theta_2} + \sum_i \frac{1}{x_i-\theta} - \sum_i \frac{1}{x_i-\theta^*} = 0 \tag{40}$$

Therefore the sum here is a purely imaginary number, as we assume that the values x_i are real. The second one of these equation shows that

$$\sum_i \frac{1}{x_i - \theta} = -\frac{1}{i\theta_2} \tag{41}$$

If we sum up here and clear the denominators, we get

$$\begin{aligned} & i\theta_2(x_1 + x_2 - 2\theta_1 - 2i\theta_2) \\ & = -x_1x_2 + (x_1 + x_2)(\theta_1 + i\theta_2) - \theta_1^2 + \theta_2^2 - 2i\theta_1\theta_2 \end{aligned} \tag{42}$$

Solving this equation shows what one already knows well about the Cauchy distribution. First, with the information of only two measured values we cannot have an estimation for the mean; it can be any value between the two measured ones. As to the variance estimator, it is also indeterminate, but this is quite a natural characteristic, so to speak, of this type of repartition, because it has no finite moments of higher order.

At this point we can easily see the advantage of Eq. (36): it shows that the best determination of the Cauchy distribution involves just as many measured values of X, as the determination of a real linear-fractional or Möbius, in terms of McCullagh, transformation. Therefore we need to have three measurements of the statistical variable X, in order to determine a Cauchy distribution the best possible way. The general estimator will then be calculated from a particularly convenient Cauchy distribution through a well-defined transformation. Let us do some calculations.

In Eqs. (40) and (41) nothing changes, except the fact that the sum should be now performed on three values of X, say x_1, x_2, x_3, instead of two. So, instead of (40) we have

$$\sum_i \frac{1}{x_i - \theta} + \sum_i \frac{1}{x_i - \theta^*} = 0; \quad \frac{3}{i\theta_2} + \sum_i \frac{1}{x_i - \theta} - \sum_i \frac{1}{x_i - \theta^*} = 0 \tag{43}$$

and instead of (41) we have

$$\sum_i \frac{1}{x_i - \theta} = -\frac{3}{2i\theta_2} \tag{44}$$

as well as the complex conjugate of this equation. Now, the direct calculation of the estimators for θ_1 and θ_2 is rather tedious. Nevertheless, we can simplify it, using the property (36), and choosing three particular values for X, say –1, 0,1, and calculate the estimator of θ for them; we then take the homographic transform of this estimator through the homography that carries –1, 0,1, into the values x_1, x_2, x_3 of X. Indeed such a real homography is well determined. Let us consider that the values (x_1, x_2, x_3) do corespond to the values (–1, 0,1) in this order. If the matrix of this homography has the entries a, b, c, d, then we can find it up to a normalization factor from the system of equations

$$x_1 = \frac{-a+b}{-c+d}; \quad x_2 = \frac{b}{d}; \quad x_3 = \frac{a+b}{c+d} \tag{45}$$

This gives

$$\frac{a}{x_2x_3 + x_1x_2 - 2x_1x_3} = \frac{b}{x_2(x_3 - x_1)} = \frac{c}{2x_2 - x_3 - x_1} = \frac{d}{x_3 - x_1} \tag{46}$$

The problem is now to find the estimator θ for the particular values (–1, 0,1). This can be easily done from Eq. (44) and its complex conjugate, which give the system

$$\theta_1 = 0; \quad 3\theta_2^2 = 1 \tag{47}$$

Therefore, in this particular case we have simply i/√(3) as an estimation for the parameter θ: it is purely imaginary. The estimator according to arbitrary data (x_1, x_2, x_3) will then be obtained through the homography given by Eq. (46):

$$\theta = \frac{(x_2x_3 + x_1x_2 - 2x_1x_3)\dfrac{i}{\sqrt{3}} + x_2(x_3 - x_1)}{(2x_2 - x_3 - x_1)\dfrac{i}{\sqrt{3}} + (x_3 - x_1)} \tag{48}$$

In real terms we have:

$$\theta_1 = \frac{\sum x_1(x_2 - x_3)^2}{\sum (x_2 - x_3)^2}; \quad \theta_2 = \sqrt{3}\,\frac{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)}{\sum (x_2 - x_3)^2} \tag{49}$$

Therefore, the complex estimator of the Cauchy distribution is in close relationship with the Hessian of the cubic having the roots (x_1, x_2, x_3). More to the point *it is the root of that Hessian*. Indeed, in terms of the roots of a cubic equation its Hessian is:

$$\left\{\sum (x_2 - x_3)^2\right\}x^2 - 2\left\{\sum x_1(x_2 - x_3)^2\right\}x + \left\{\sum x_1^2(x_2 - x_3)^2\right\} = 0 \tag{50}$$

and its roots are θ above and its complex conjugate. The expression from Eq. (49) are the real and imaginary parts of these roots. Even more, the sum and the product of the two complex estimators are given by the mean and the standard deviation of the three values, with respect to the system of probabilities:

$$p_1 \equiv \frac{(x_2 - x_3)^2}{\sum (x_2 - x_3)^2}, \quad p_2 \equiv \frac{(x_3 - x_1)^2}{\sum (x_2 - x_3)^2}, \quad p_3 \equiv \frac{(x_1 - x_2)^2}{\sum (x_2 - x_3)^2} \tag{51}$$

which they determine quite naturally.

In our statistical interpretation, the root of the Hessian is the parameter of a Cauchy distribution. The roots of the corresponding cubic are three measurements of the Cauchy variate, that give the most reliable estimate of the parameter. The problem with this representation of the connection between cubic and its Hessian is that the Cauchy distribution is referring to a one-dimensional variate. However, everything gets in order if we take the Cauchy density of probability as a marginal distribution of a Gaussian in plane.

9. Conclusions

The main conclusions of the present paper are the following:

i) Some meanings and notations for every binary cubic are presented;

ii) The algebra of the cubic equation is formulated;

iii) In the algebra of the cubic equation, an external gauge factor is established and a physical interpretation is given;

iv) Our physical interpretation leads us to Novozhilov's statistics and the specification of Hessian;

v) Concentrating on the pure mathematical side of the problem, a Barbilian differential geometry is obtained;

vi) theA correlation between the apolar transport of cubics and a Levy-Civita parallel transport in Lobachevsky's plane are established;

vii) Peter McCullagh's observations on Cauchy Statistics are discussed;

viii) All our findings can be applied to other non-linear models, such as the well-known Kuznetsov (Kuznetsov *et al*., 1994) model for tumor growth.

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ASUPRA UNEI GEOMETRII FIZICE ȘI NOI CONEXIUNI STATISTICE

(Rezumat)

În prezenta lucrare se descrie acțiunea unei matrici reale 2x2. Este o acțiune simplă tranzitivă, ce caracterizează un etalon propriu, conectat la matrici ce pot reprezenta tensiuni ale unui mediu continuu. Geometria compatibilă cu acest etalon este o generalizare tridimensională a geometriei hiperbolice plane. În termeni reali, această geometrie este una de tip Lorentz. Mai mult, sunt prezentate interpretări fizice ale acestui model și, în special, pentru unghiul etalon. Poate fi observată o conexiune puternică între acest model și cel al lui Kuznetsov privind creșterea tumorală.